

A condition on the chiral symmetry breaking solution of the Dyson-Schwinger equation in three-dimensional QED

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Abstract

In three-dimensional QED, which is analyzed in the $1/N$ expansion, we obtain a sufficient and necessary condition for a nontrivial solution of the Dyson-Schwinger equation to be chiral symmetry breaking solution. In the derivation, a normalization condition of the Goldstone bound state is used. It is showed that the existent analytical solutions satisfy this condition.

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The interest in quantum electrodynamics in two space and one time dimensions (QED_3) is twofold. On the one hand, QED_3 itself as a quantum field theory exhibits many interesting features, such as dynamical chiral symmetry breaking (DCSB) [1-5] and confinement [6], which also exist in QCD_4 and other gauge theories. Furthermore, unlike four dimensional QED, QED_3 has a dimensional coupling constant e and therefore is a superrenormalizable field theory, which makes it easier to be treated. It is expected that the study on QED_3 would shed light on our understandings of more complicated gauge theories like QCD. On the other hand, QED_3 has been used to describe some planar condensed matter systems, especially high temperature cuprates. It is well established that an undoped cuprate is a 2d quantum antiferromagnet and described by the spin-1/2 Heisenberg model [7], which can be mapped into a theory of massless fermions coupled to a $U(1)$ gauge field [8]. The chiral symmetry breaking in this $U(1)$ gauge theory corresponds to the Néel ordering in the antiferromagnet [8,9].

One of the most interesting characteristics of QED_3 is that it may exhibit DCSB, from which a massless fermion acquires a dynamically generated mass. This is a nonperturbative effect and conventional perturbative expansion approach is unable to investigate this problem. The standard mathematical tool of analyzing chiral phase transition is the Dyson-Schwinger (DS) nonlinear integral equation for the fermion self-energy. If the DS equation has only vanishing solution, the theory is chirally symmetric and the fermions remain massless. The case of interest to us is when a nontrivial solution of the DS equation occurs. Generally, the existence of a nontrivial solution of the DS equation was believed to lead unambiguously to the existence of DCSB. However, as pointed out by Cheng and Kuo [10], where they discussed this problem in quenched planar QED_4 , this is not always the case. Actually, the

breaking of a global chiral symmetry is always accompanied by a (only one in U(1) gauge theory) massless Goldstone boson [11], which is a pseudoscalar bound state composed of a fermion and an antifermion [12]. Once the DS equation develops a nontrivial solution, there should also be a nontrivial solution for the corresponding Bethe-Salpeter (BS) equation satisfied by this bound state wave function. Note that: these solutions are not independent and can be converted into each other under a simple transformation, and the BS wave function must satisfy an additional normalization condition to ensure the bound state is stable [13]. Using these two facts, we may obtain a constraint on the nontrivial solution of the DS equation from the normalization condition of the BS wave function. Therefore, that the DS equation has a nontrivial solution is only a necessary but not a sufficient condition to lead to DCSB.

In the spirit of the above discussions, Cheng and Kuo [10] have considered the quenched planar QED₄ and acquired a necessary and sufficient condition for a nontrivial solution of the DS equation to be symmetry breaking. They applied this condition to the explicit solution of the DS equation given by Kondo *et al.* [14], and found that, although DCSB occurs when there is a cutoff, once the cutoff is taken away DCSB disappears.

In this paper, we examine this problem in the continuous QED₃ and verify whether or not the existed nontrivial solutions of the DS equations lead to DCSB. The Lagrangian of QED₃ in Euclidean space is

$$L = \sum_{i=1}^N \bar{\psi}_i (i\partial_\mu + eA_\mu) \gamma_\mu \psi_i + \frac{1}{4} F_{\mu\nu}^2, \quad (1)$$

with N fermions which are four-component spinors. In this case, the 4×4 γ_3 and γ_5 matrices can be well constructed, which anticommute with γ_0 , γ_1 ,

and γ_2 . For details about the γ matrices, see Ref. [2]. The Lagrangian is invariant under the chiral transformations $\psi \rightarrow \exp(i\theta\gamma_3)\psi$ and $\psi \rightarrow \exp(i\omega\gamma_5)\psi$ because the fermions are massless. A mass term $m\bar{\psi}\psi$, no matter it is added to the Lagrangian by hand or dynamically generated, would break these symmetries. Since the mechanism of dynamical mass generation for fermions needs no additional Higgs particles, it is very interesting to study DCSB in various field theories.

Power-counting arguments show that the square of the coupling constant e^2 has dimensions of mass, therefore QED₃ is a superrenormalizable field theory and ultraviolet (UV) finite. However, while the UV behavior becomes better, in the massless case, perturbative expansions in e^2 lead to infrared (IR) divergences which appear already at two loops. One way to overcome this difficulty is using the $1/N$ expansion [15].

We write the full fermion propagator as

$$S^{-1}(p) = i\gamma \cdot p A(p^2) + B(p^2), \quad (2)$$

where $A(p^2)$ is the wave-function renormalization and $B(p^2)$ is the fermion self-energy. The DS equation for the full fermion propagator [16] is

$$S^{-1}(p) = S_0^{-1}(p) - e^2 \int \frac{d^3q}{(2\pi)^3} \gamma_\mu S(q) \Gamma_\nu(p, q) D_{\mu\nu}(p - q), \quad (3)$$

where $\Gamma_\nu(p, q)$ is the full vertex function and the full photon propagator is

$$D_{\mu\nu}(q) = \left(\delta_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) \frac{1}{q^2 [1 + \Pi(q^2)]} + \xi \frac{q_\mu q_\nu}{q^4}, \quad (4)$$

with the vacuum polarization $\Pi(q^2)$ related to the vacuum polarization tensor $\Pi_{\mu\nu}(q)$ as follows:

$$\Pi_{\mu\nu}(q) = (q^2\delta_{\mu\nu} - q_\mu q_\nu)\Pi(q^2). \quad (5)$$

Substituting Eq.(2) into Eq.(3) and taking trace on both sides, we obtain the equation for $B(p^2)$

$$B(p^2) = -\frac{e^2}{4} \int \frac{d^3q}{(2\pi)^3} \text{tr}[\gamma_\mu S(q)\Gamma_\nu(p, q)D_{\mu\nu}(p - q)]. \quad (6)$$

Multiplying both sides of Eq.(3) by $\gamma \cdot p$ and then taking trace on both sides, we obtain the equation for $A(p^2)$

$$A(p^2) = 1 + \frac{e^2}{4p^2} \int \frac{d^3q}{(2\pi)^3} \text{tr}[(i\gamma \cdot p)\gamma_\mu S(q)\Gamma_\nu(p, q)D_{\mu\nu}(p - q)]. \quad (7)$$

In this paper, in order to satisfy the Ward-Takahashi identity, the vertex function [17] is taken to be

$$\Gamma_\mu(p, q) = \gamma_\mu G(p^2, q^2). \quad (8)$$

The one-loop vacuum polarization [2] is

$$\Pi(p^2) = \alpha / |p| \quad (9)$$

with $\alpha = Ne^2/8$.

Now, we can write the equation for $B(p^2)$ as:

$$B(p^2) = -\frac{e^2}{4} \int \frac{d^3q}{(2\pi)^3} \frac{B(q^2)G(p^2, q^2)}{q^2 A^2(q^2) + B^2(q^2)} \text{tr}[\gamma_\mu \gamma_\nu D_{\mu\nu}(p - q)]. \quad (10)$$

To look for DCSB in QED₃, the nonlinear integral equations for $A(p^2)$ and $B(p^2)$ should be solved explicitly. It is easy to see that there is always a trivial solution $B(p^2) \equiv 0$, which means there is no mass generation of fermions and hence no DCSB. The conventional opinion claimed that a non-trivial solution $B(p^2)$ leads to DCSB. However, according to the well-known

Goldstone theorem [11], the spectrum of physical particles must contain one massless spin-0 particle for each broken continuous symmetry. If the broken symmetry is a global chiral symmetry, the corresponding Goldstone boson is a fermion-antifermion (A and \bar{A}) pseudoscalar bound state. When we study DCSB, the properties of the Goldstone boson must be taken into account. We have seen in Ref. [10] that the normalization condition of this bound state places a nontrivial constraint on the asymptotic form of $A(p^2)$ and $B(p^2)$ in quenched planar QED₄. Now, we address this constraint in the three-dimensional QED.

This bound state is a nontrivial solution of the following BS equation in Euclidean space [18]:

$$S_A^{-1}\left(\frac{k}{2} + p\right)\chi_k(p)S_{\bar{A}}^{-1}\left(\frac{k}{2} - p\right) = -e^2 \int \frac{d^3q}{(2\pi)^3} \gamma_\mu \chi_k(q) \Gamma_\nu(p, q) D_{\mu\nu}(p - q). \quad (11)$$

Here, $k/2 = (p_A + p_{\bar{A}})/2$ and $p = (p_A - p_{\bar{A}})$ are the center-of-mass 3-momentum and relative 3-momentum of the bound state, respectively. $\chi_k(p)$ is the bound state wave function expressed as a 4×4 matrix.

Combining Eq.(2) and Eq.(11), we have

$$\begin{aligned} & \left[i\gamma \cdot \left(\frac{k}{2} + p\right) A \left(\left(\frac{k}{2} + p\right)^2 \right) + B \left(\left(\frac{k}{2} + p\right)^2 \right) \right] \\ & \times \chi_k(p) \left[i\gamma \cdot \left(\frac{k}{2} - p\right) A \left(\left(\frac{k}{2} - p\right)^2 \right) - B \left(\left(\frac{k}{2} - p\right)^2 \right) \right] \\ & = -e^2 \int \frac{d^3q}{(2\pi)^3} \gamma_\mu \chi_k(q) \Gamma_\nu(p, q) D_{\mu\nu}(p - q) \end{aligned} \quad (12)$$

Note that since we adopt four-component spinors, there are two matrices which anticommute with γ_0 , γ_1 and γ_2 . The mass term $m\bar{\psi}\psi$ breaks two chiral symmetries simultaneously. As a result there are two Goldstone bosons respectively coupling to $\bar{\psi}\gamma_\mu\gamma_3\psi$ and $\bar{\psi}\gamma_\mu\gamma_5\psi$. These Goldstone bound states

both have the quantum number $l^p = 0^-$, which corresponds to the broken generator of the chiral group and vanishing 3-momentum. Their BS amplitudes are $\chi_k(p) = \chi_{03}(p^2)\gamma_3$ and $\chi_k(p) = \chi_{05}(p^2)\gamma_5$ respectively with $k = 0$. It is easy to show that $\chi_{03}(p^2)$ and $\chi_{05}(p^2)$ satisfy the same BS equation and normalization condition, which give the same constraint on the nontrivial solution of the DS equation. For simplicity, we take γ_5 as the example and the result remains correct for γ_3 .

Substituting $\chi_k(p) = \chi_{05}(p^2)\gamma_5$ and $k = 0$ into Eq.(12), and taking trace on both sides, we finally obtain the BS equation for the Goldstone boson:

$$[p^2 A^2(p^2) + B^2(p^2)]\chi_{05}(p^2) = -\frac{e^2}{4} \int \frac{d^3 q}{(2\pi)^3} \chi_{05}(q^2) G(p^2, q^2) \text{tr}[\gamma_\mu \gamma_\nu D_{\mu\nu}(p-q)], \quad (13)$$

where $A(p^2)$ and $B(p^2)$ are solutions of Eq.(6) and Eq.(7) respectively. In the derivation of the right-hand side of this equation, we have used the approximation $\Gamma_\nu(p, q) = \gamma_\nu G(p^2, q^2)$.

The BS equation (13) is a homogeneous integral equation and hence can not determine the bound state completely, leaving an arbitrary multiplicative finite constant C . To eliminate this uncertainty, an additional normalization condition of $\chi_{05}(p^2)$ is necessary. The search for such a normalization condition has a long history, and there are several approaches (different but equivalent) to normalize the bound state wave function [13]. In this paper, for convenience, we shall utilize the form given by Suttrop [19]. Rewriting the normalization condition in three dimension, we have

$$\begin{aligned} & \int d^3 p \text{tr} \left\{ \bar{\chi}_k(p) \left[i\gamma \cdot \left(\frac{k}{2} + p \right) A \left(\left(\frac{k}{2} + p \right)^2 \right) + B \left(\left(\frac{k}{2} + p \right)^2 \right) \right] \right. \\ & \left. \times \chi_k(p) \left[i\gamma \cdot \left(\frac{k}{2} - p \right) A \left(\left(\frac{k}{2} - p \right)^2 \right) - B \left(\left(\frac{k}{2} - p \right)^2 \right) \right] \right\} \end{aligned}$$

$$= \lambda \frac{dM_B}{d\lambda} \quad (14)$$

where M_B is the mass of the bound state and $\lambda = e$ is the coupling constant. Generally, it is sufficient to require that the integral at the left-hand side of Eq.(14) has a finite value [19]. Setting $k = 0$, $\chi_k(p) = \chi_{05}(p^2)\gamma_5$ and using the identity $\gamma_\mu\gamma_5 = -\gamma_5\gamma_\mu$, the normalization condition can be written in the simple form

$$\int_0^\infty dq q^2 [q^2 A^2(q^2) + B^2(q^2)] \chi_{05}^2(q^2) = finite. \quad (15)$$

Now we introduce the function

$$\phi(p^2) = [p^2 A^2(p^2) + B^2(p^2)] \chi_{05}(p^2), \quad (16)$$

then Eq.(13) becomes

$$\phi(p^2) = -\frac{e^2}{4} \int \frac{d^3 q}{(2\pi)^3} \frac{\phi(q^2)}{q^2 A^2(q^2) + B^2(q^2)} G(p^2, q^2) tr[\gamma_\mu \gamma_\nu D_{\mu\nu}(p - q)]. \quad (17)$$

If there is a solution $B(p^2)$ of Eq.(10) (the equation for $A(p^2)$ always has a solution), then we also have a solution of Eq.(17) as $\phi(p^2) = cB(p^2)$, with c an arbitrary finite constant. According to Eq.(16), the Goldstone boson $\chi_{05}(p^2)$ is given by $\phi(p^2)$, $A(p^2)$ and $B(p^2)$. Thus, we have constructed a relationship between two systems, one is described by DS equation and the other BS equation. Therefore, Eq.(16) and Eq.(17) may be regarded as another form of the Goldstone theorem.

Using Eq.(16), the normalization condition Eq.(15) can be written as

$$\int_0^\infty dq \frac{q^2 \phi^2(q^2)}{q^2 A^2(q^2) + B^2(q^2)} = finite. \quad (18)$$

For chiral symmetry to be broken, the nontrivial solution of the DS equation (10) and the BS equation (13) must exist simultaneously. Besides, as

a bound state wave function, each nontrivial solution of the BS equation satisfies a normalization condition. We can easily turn this condition into a constraint placed on the nontrivial solution of the DS equation, and now state a theorem.

Theorem: The necessary and sufficient condition for a nontrivial solution $B(p^2)$ to be chiral symmetry breaking solution is that it must satisfy, together with the solution $A(p^2)$, the condition

$$\int_0^\infty dq \frac{q^2 B^2(q^2)}{q^2 A^2(q^2) + B^2(q^2)} = \text{finite}. \quad (19)$$

The proof of a similar theorem in the case of QED₄ given in Ref. [10] is still valid in the current theory although the conditions in these two cases are different. Hence, we will not present the proof in this paper.

It is now straightforward to apply this result to the nontrivial solution of the DS equation. Although the approximations (Eq.(8) and Eq.(9)) that we take in this paper simplify the DS equation significantly, it appears impossible to obtain its completely analytical solution. To solve Eq.(10) analytically, further approximations are needed. Under the bare vertex and one loop vacuum-polarization approximations, Nash [4] considered the leading and next-to-leading terms in the $1/N$ expansion of the kernels in the DS equation and showed that higher-order corrections do not alter the nature of the symmetry breaking. This result confirms the qualitative conclusion made earlier by Appelquist and coworkers [3] that there is a finite critical number of flavors N_c above which the DS equation for $B(p^2)$ has no nontrivial solutions. Nash solved the DS equation for $B(p^2)/A(p^2)$ in low momentum region and showed that N_c is gauge-independent. For $N < N_c$, considering

only first order of $1/N$ in this paper, $B(p^2)/A(p^2)$ has the form

$$\frac{B(p^2)}{A(p^2)} = p^{-\frac{1}{2}} \sin \left(\frac{1}{2} \left[\frac{128}{3\pi^2 N} - 1 \right]^{\frac{1}{2}} \left\{ \ln \left[\frac{pA(0)}{B(0)} + \delta \right] \right\} \right), \quad (20)$$

in the infrared region, where δ is a phase and $B(0)/A(0)$ is finite. In the ultraviolet region ($p \gg \alpha$), the mass function $B(p^2)/A(p^2)$ falls like $1/p^2$ [2]. This result was obtained via the operator-product expansion approach and does not depend on the $1/N$ expansion [2].

In the current case, the condition (19) becomes

$$\int_0^\infty dq \frac{q^2}{1 + q^2 A^2(q^2)/B^2(q^2)} = finite. \quad (21)$$

It is easy to see that there are no singularities in the integrand. The dangers those would cause Eq.(21) invalid come from the ultraviolet and infrared behaviors of $B(p^2)/A(p^2)$. Direct calculations show that the ultraviolet and infrared forms of $B(p^2)/A(p^2)$ satisfy this condition. Thus, at least to the lowest order in $1/N$ expansion, the nontrivial solution of the DS equation in QED₃ does lead to DCSB and dynamical mass generation for fermions.

In conclusion, we have obtained a sufficient and necessary condition for a nontrivial solution of the DS equation to be chiral symmetry breaking solution. In the derivation of this condition, the Goldstone theorem and the normalization condition of the Goldstone bound state wave function play important roles. It is showed that the nontrivial solutions given by Nash and Appelquist *et al.* [2] satisfy this condition. Therefore we see that QED in four dimension (in quenched planar approximation) and three dimension (under the approximations mentioned above) have different chiral phase structures. The former undergoes chiral phase transition only when a cutoff is present, but the latter exhibits DCSB in the continuum form when the number of

fermions is less than a critical value. The origin of this difference is still unknown and subjected to the future studies.

It is claimed [4] that the intrinsic scale α divides QED₃ into two parts and everything in high energy beyond α is rapidly damped. From this point of view, it seems reasonable to construct an equivalence between the continuous field theory, QED₃, and the low-energy effective theory of antiferromagnet defined on discrete lattices. In some sense, α is the lattice constant of QED₃ [8].

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